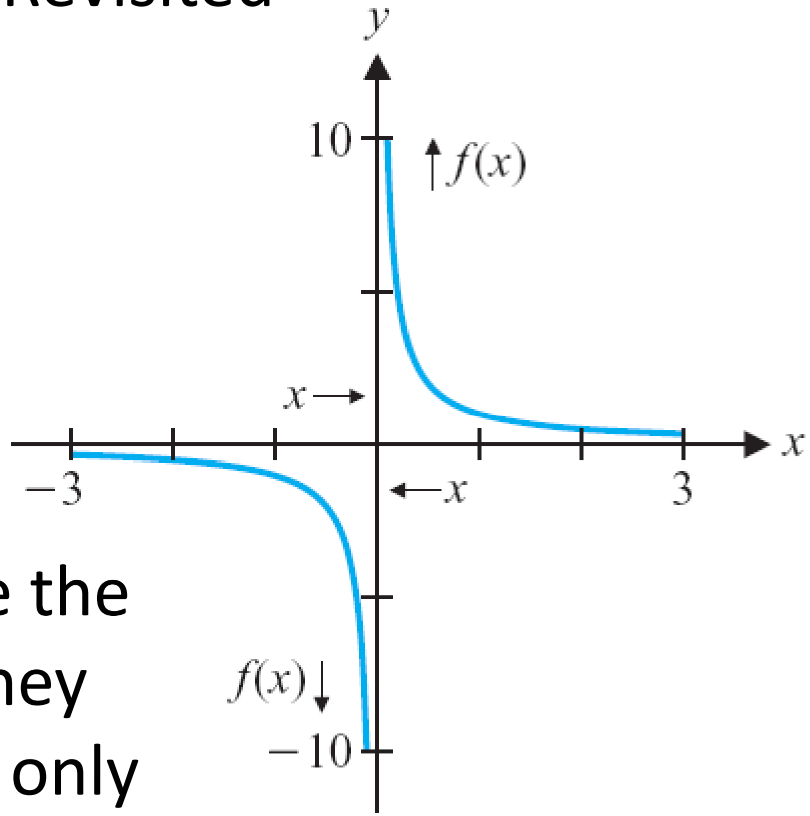
EXAMPLE 5.1 A Simple Limit Revisited

$$\text{Examine } \lim_{x \to 0} \frac{1}{x}.$$

EXAMPLE 5.1 A Simple Limit Revisited



Solution

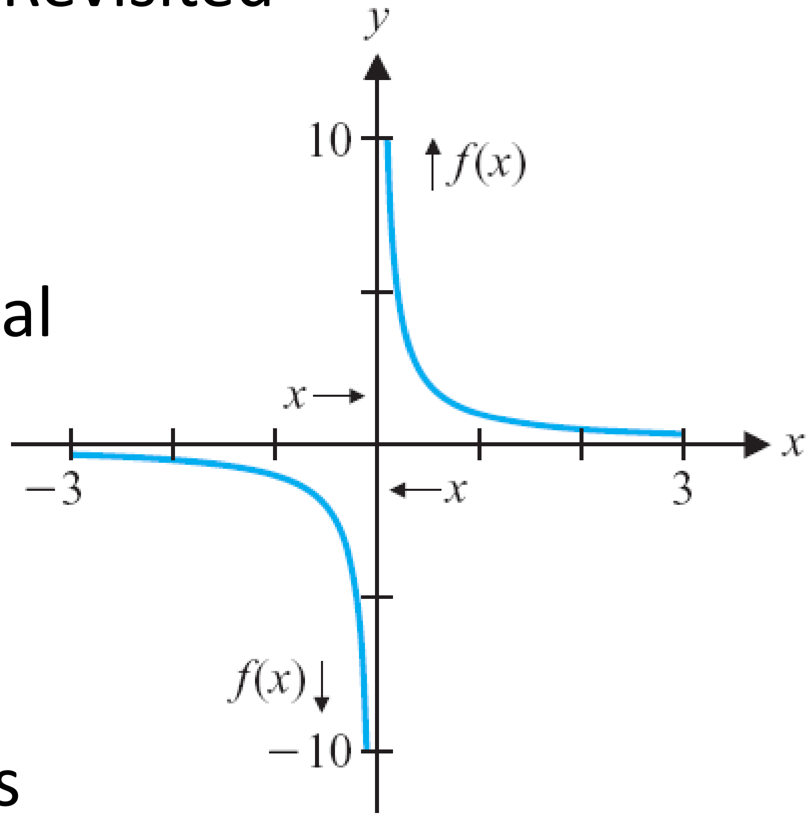
We write

$$\lim_{x\to 0^+} \frac{1}{x} = \infty \qquad \lim_{x\to 0^-} \frac{1}{x} = -\infty$$

It is important to note that while the limits *do not exist,* we say that they "equal" ∞ and −∞, respectively, only to be specific as to *why* they do not exist.

1.5 LIMITS INVOLVING INFINITY; ASYMPTOTES

EXAMPLE 5.1 A Simple Limit Revisited



Solution

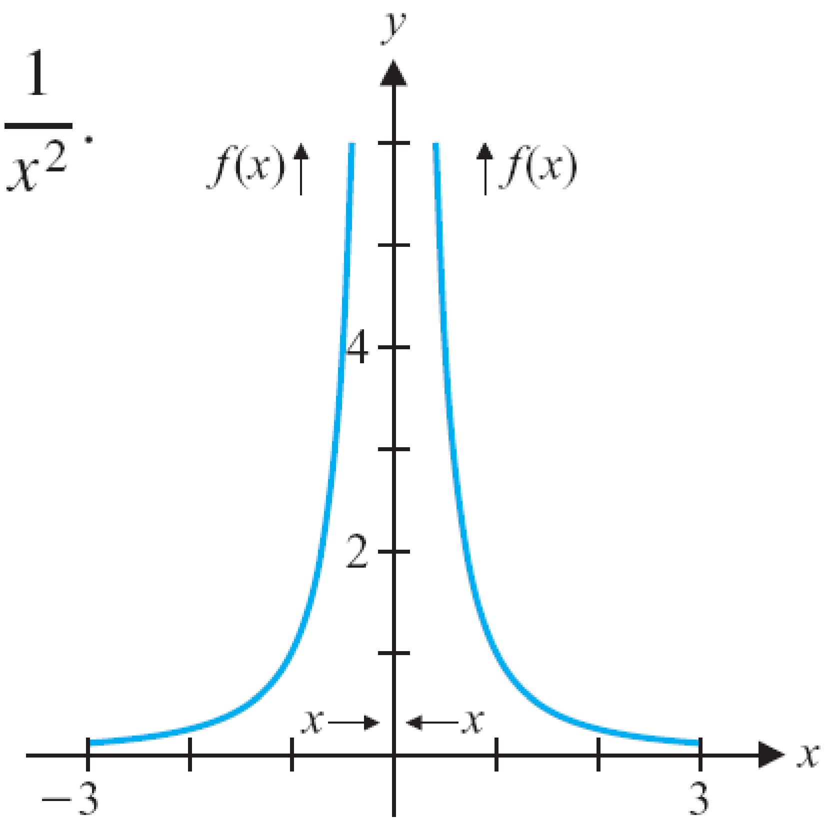
The graph approaches the vertical line $x = 0$ as $x \to 0$.

We say that the line $x = 0$ is a **vertical asymptote.**

Finally, since the one-sided limits do not agree, we say $\lim_{x\to 0} \frac{1}{x}$ does not exist

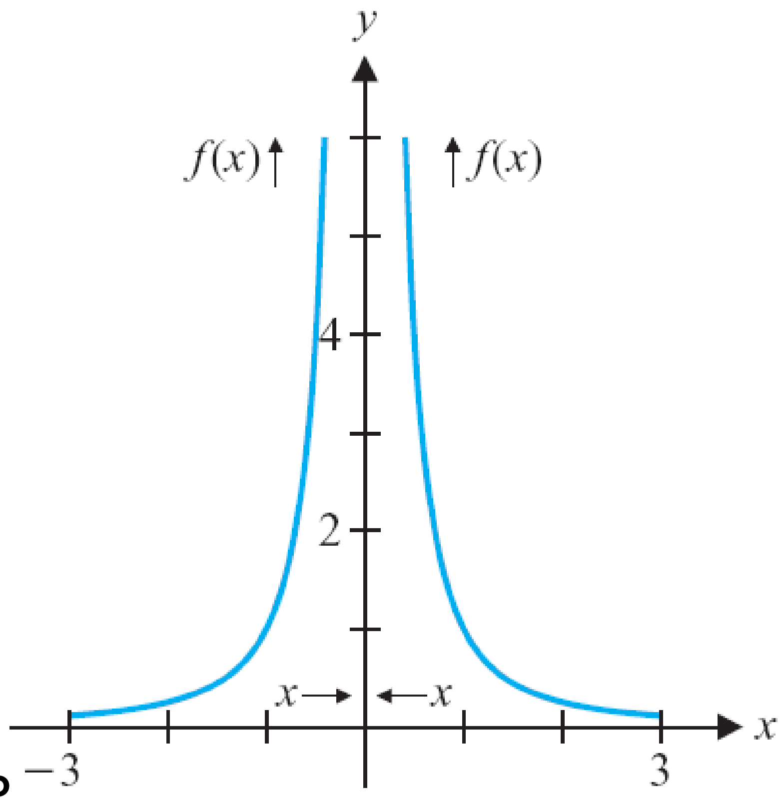
EXAMPLE 5.2 A Function Whose One-Sided Limits Are Both Infinite

Evaluate $\lim_{x \to 0} \frac{1}{x^2}$.



1.5 LIMITS INVOLVING INFINITY; ASYMPTOTES

EXAMPLE 5.2 A Function Whose One-Sided Limits Are Both Infinite



Solution

Since both one-sided limits agree (i.e., both tend to ∞), we say that

$$\lim_{x \to 0} \frac{1}{x^2} = \infty.$$

This one concise statement says that the limit does not exist, but also that there is a vertical asymptote at $x = 0$, where $f(x) \to \infty$ as $x \to 0$ from either side.

1.5 LIMITS INVOLVING INFINITY; ASYMPTOTES

REMARK 5.1

It may at first seem contradictory to say that $\lim_{x \to 0^+} \frac{1}{x}$ does not exist and then to write

$$\lim_{x \to 0^+} \frac{1}{x} = \infty.$$

However, since ∞ is *not* a real number, there is no contradiction here. We say that

$$\lim_{x \to 0^+} \frac{1}{x} = \infty$$

to indicate that as $x \to 0^+$, the function values are increasing without bound.

1.5 LIMITS INVOLVING INFINITY; ASYMPTOTES

REMARK 5.2

Mathematicians try to convey as much information as possible with as few symbols as possible. For instance,

we prefer to say $\lim_{x\to 0} \frac{1}{x^2} = \infty$ rather than $\lim_{x\to 0} \frac{1}{x^2}$

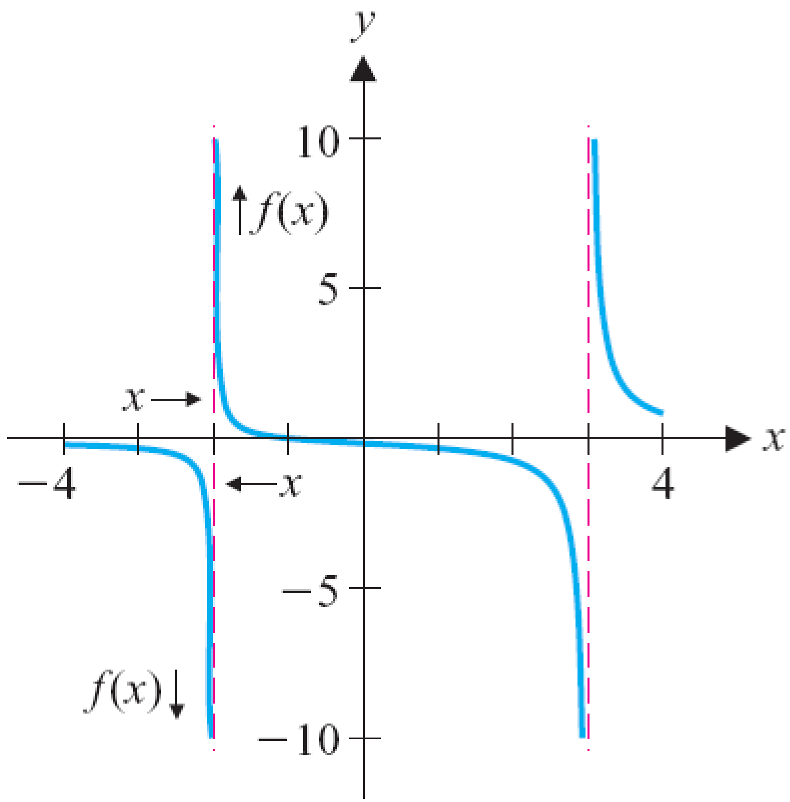
does not exist, since the first statement not only says that the limit does not exist, but also says that increases without bound as x approaches 0, with $x > 0$ or $x < 0$.

EXAMPLE 5.4 Another Case Where Infinite One-Sided Limits Disagree

Evaluate $\lim_{x \to -2} \frac{x+1}{(x-3)(x+2)}$.

1.5 LIMITS INVOLVING INFINITY; ASYMPTOTES

EXAMPLE 5.4 Another Case Where Infinite One-Sided Limits Disagree



Solution

There appears to be a vertical asymptote at $x = -2$. Further, the function appears to tend to ∞ as $x \to -2^+$, and to $-\infty$ as $x \to -2^-$.

1.5 LIMITS INVOLVING INFINITY; ASYMPTOTES

EXAMPLE 5.4 Another Case Where Infinite One-Sided Limits Disagree

Solution Verify the signs of the one-sided limits:

$$\lim_{x\to -2^+} \frac{\overset{-}{x+1}}{\underset{-}{(x-3)}\underset{+}{(x+2)}} = \infty$$

Since $(x+1) < 0$, $(x-3) < 0$ and $(x+2) > 0$, for $-2 < x < -1$.

$$\lim_{x\to -2^-} \frac{\overset{-}{x+1}}{\underset{-}{(x-3)}\underset{-}{(x+2)}} = -\infty$$

Since $(x+1) < 0$, $(x-3) < 0$ and $(x+2) < 0$, for $x < -2$.

There is indeed a vertical asymptote at $x = -2$ and

$$\lim_{x\to -2} \frac{x+1}{(x-3)(x+2)} \text{ does not exist.}$$

1.5 LIMITS INVOLVING INFINITY; ASYMPTOTES

Limits at Infinity

We are also interested in examining the limiting behavior of functions as x increases without bound (written $x \to \infty$) or as x decreases without bound (written $x \to -\infty$).

1.5 LIMITS INVOLVING INFINITY; ASYMPTOTES

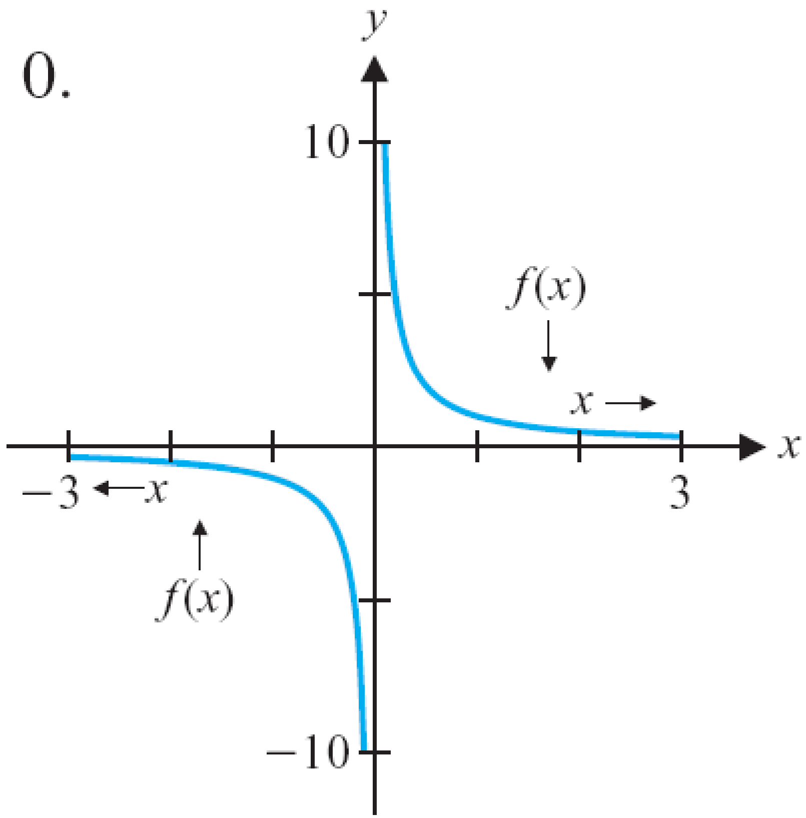
- Limits at Infinity

Returning to $f(x) = \dfrac{1}{x}$, we can see that as $x \to \infty, \dfrac{1}{x} \to 0.$

In view of this, we write $\lim\limits_{x \to \infty} \dfrac{1}{x} = 0.$

Similarly, $\lim\limits_{x \to -\infty} \dfrac{1}{x} = 0.$

The graph appears to approach the horizontal line $y = 0$, as $x \to \infty$ and as $x \to -\infty$. We call $y = 0$ a **horizontal asymptote.**



1.5 LIMITS INVOLVING INFINITY; ASYMPTOTES

EXAMPLE 5.6 Finding Horizontal Asymptotes

Find any horizontal asymptotes to the graph of $f(x) = 2 - \frac{1}{x}$.

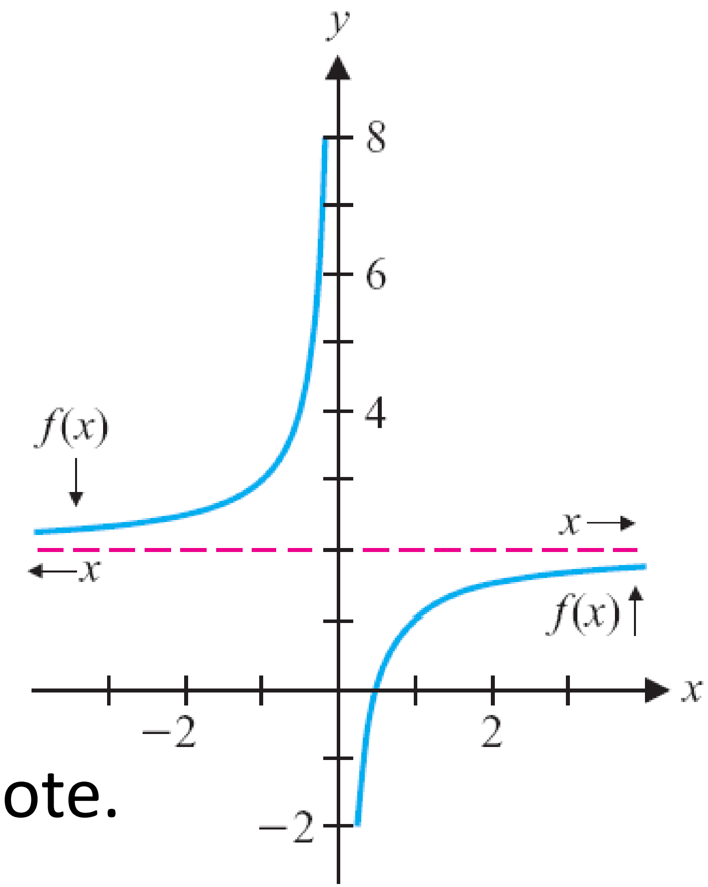
EXAMPLE 5.6 Finding Horizontal Asymptotes

Solution

As $x \to \pm\infty, \frac{1}{x} \to 0$

So, $$\lim_{x \to \infty} \left(2 - \frac{1}{x}\right) = 2$$

$$\lim_{x \to -\infty} \left(2 - \frac{1}{x}\right) = 2$$



The line $y = 2$ is a horizontal asymptote.

THEOREM 5.1

For any rational number $t > 0$,

$$\lim_{x \to \pm\infty} \frac{1}{x^t} = 0,$$

where for the case where $x \to -\infty$, we assume that

$$t = \frac{p}{q}, \text{ where } q \text{ is odd.}$$

1.5 LIMITS INVOLVING INFINITY; ASYMPTOTES

THEOREM 5.2

For a polynomial of degree $n > 0$,

$$p_n(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0,$$

we have

$$\lim_{x\to\infty} p_n(x) = \begin{cases} \infty, & \text{if } a_n > 0 \\ -\infty, & \text{if } a_n < 0 \end{cases}.$$

Observe that you can make similar statements regarding the value of $\lim_{x\to-\infty} p_n(x)$, but be careful: the answer will change depending on whether *n* is even or odd. (We leave this as an exercise.)

EXAMPLE 5.7 A Limit of a Quotient That Is Not the Quotient of the Limits

Evaluate $\lim_{x \to \infty} \dfrac{5x - 7}{4x + 3}$.

EXAMPLE 5.7 A Limit of a Quotient That Is Not the Quotient of the Limits

Solution

You might be tempted to write

$$\lim_{x\to\infty}\frac{5x-7}{4x+3}=\frac{\lim\limits_{x\to\infty}(5x-7)}{\lim\limits_{x\to\infty}(4x+3)}$$

This is an incorrect use of Theorem 3.1, since the limits in the numerator and the denominator do not exist.

$$=\frac{\infty}{\infty}=1.$$

This is incorrect!

EXAMPLE 5.7 A Limit of a Quotient That Is Not the Quotient of the Limits

Solution

When a limit has the form ∞/∞, the actual value of the limit can be anything at all.

For this reason, we call ∞/∞ an **indeterminate form,** meaning that the value of the limit cannot be determined solely by noticing that both numerator and denominator tend to ∞.

EXAMPLE 5.7 A Limit of a Quotient That Is Not the Quotient of the Limits

Solution

Rule of Thumb: When faced with the indeterminate form ∞/∞ in calculating the limit of a rational function, divide numerator and denominator by the highest power of x appearing in the *denominator.*

1.5 LIMITS INVOLVING INFINITY; ASYMPTOTES

EXAMPLE 5.7 A Limit of a Quotient That Is Not the Quotient of the Limits

Solution

$$\lim_{x\to\infty} \frac{5x-7}{4x+3} = \lim_{x\to\infty} \left[\frac{5x-7}{4x+3} \cdot \frac{(1/x)}{(1/x)}\right]$$ Multiply numerator and denominator by $\frac{1}{x}$.

$$= \lim_{x\to\infty} \frac{5-7/x}{4+3/x}$$ Multiply through by $\frac{1}{x}$.

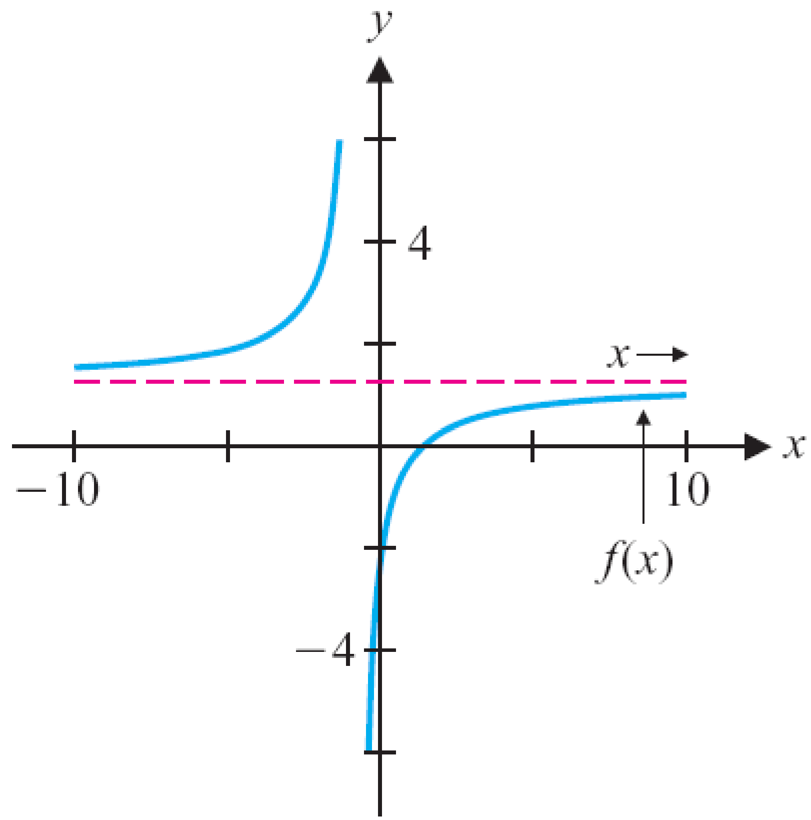
$$= \frac{\lim_{x\to\infty}(5-7/x)}{\lim_{x\to\infty}(4+3/x)}$$ By Theorem 3.1 (iv).

$$= \frac{5}{4} = 1.25,$$

EXAMPLE 5.7 A Limit of a Quotient That Is Not the Quotient of the Limits

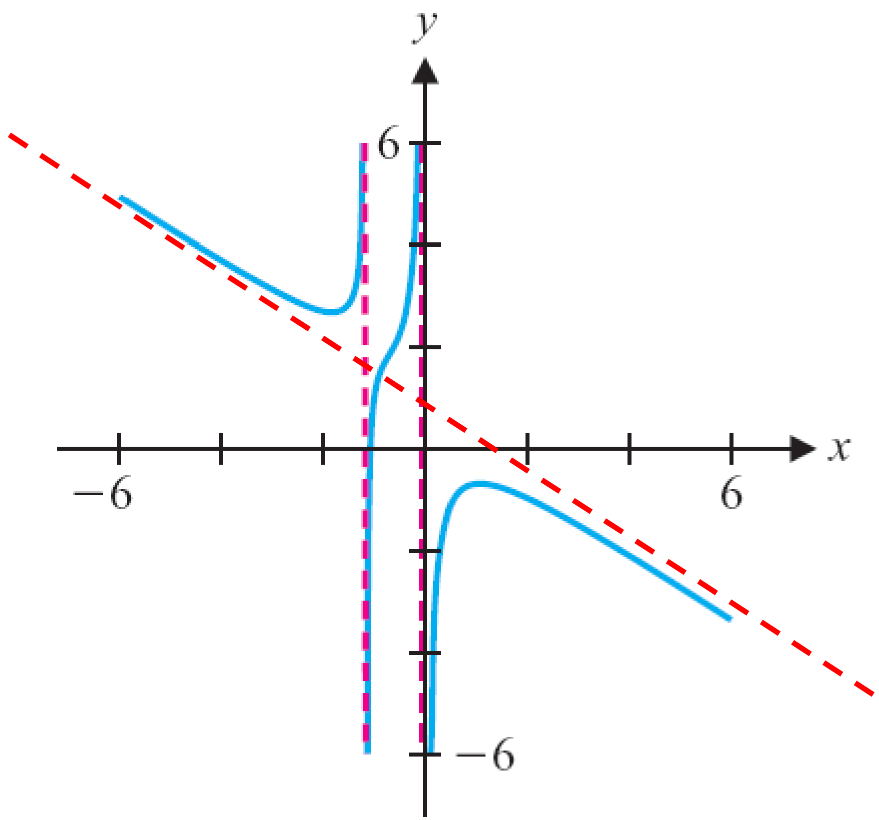
Solution

$$\lim_{x\to\infty} \frac{5x-7}{4x+3} = \frac{5}{4} = 1.25$$



EXAMPLE 5.8 Finding Slant Asymptotes

Evaluate $\lim_{x\to\infty} \dfrac{4x^3+5}{-6x^2-7x}$ and find any slant asymptotes.



EXAMPLE 5.8 Finding Slant Asymptotes

Solution Using our rule of thumb, we have

$$\lim_{x\to\infty} \frac{4x^3+5}{-6x^2-7x} = \lim_{x\to\infty} \left[\frac{4x^3+5}{-6x^2-7x} \cdot \frac{(1/x^2)}{(1/x^2)}\right] \quad \text{Multiply numerator and denominator by } \frac{1}{x^2}.$$

$$= \lim_{x\to\infty} \frac{4x+5/x^2}{-6-7/x} \quad \text{Multiply through by } \frac{1}{x^2}.$$

$$= -\infty,$$

EXAMPLE 5.8 Finding Slant Asymptotes

Solution Performing long division:

$$\frac{4x^3+5}{-6x^2-7x} = -\frac{2}{3}x + \frac{7}{9} + \frac{5+49/9x}{-6x^2-7x}$$

The third term tends to 0 as $x \to \infty$, so the function values approach those of the linear function

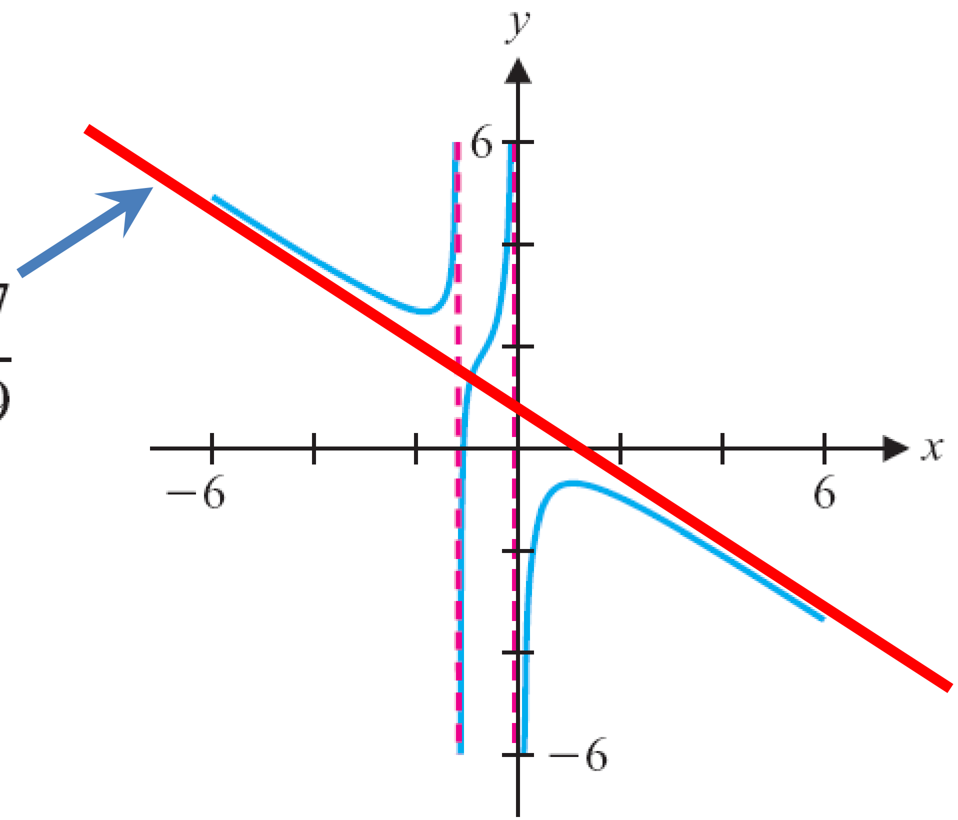
$$y = -\frac{2}{3}x + \frac{7}{9}$$

For this reason, we say that the graph has a **slant (or oblique) asymptote.**

EXAMPLE 5.8 Finding Slant Asymptotes

Solution

$$y = -\frac{2}{3}x + \frac{7}{9}$$



EXAMPLE 5.9 Two Limits of an Exponential Function

Evaluate $\lim_{x \to 0^-} e^{1/x}$ and $\lim_{x \to 0^+} e^{1/x}$.

EXAMPLE 5.9 Two Limits of an Exponential Function

Solution

$$\lim_{x\to 0^-} \frac{1}{x} = -\infty \text{ and } \lim_{x\to -\infty} e^x = 0$$

$$\rightarrow \lim_{x\to 0^-} e^{1/x} = 0$$

$$\lim_{x\to 0^+} \frac{1}{x} = \infty \text{ and } \lim_{x\to \infty} e^x = \infty$$

$$\rightarrow \lim_{x\to 0^+} e^{1/x} = \infty$$